Oxford Cambridge and RSA

Monday 23 June 2025 – Afternoon

A Level Further Mathematics B (MEI)

Y436/01 Further Pure with Technology

Time allowed: 1 hour 45 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a computer with appropriate software
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

1 A family of curves is given by the cartesian equation

$$\frac{x^2}{a^2}+mxy+\frac{y^2}{b^2}=1$$

where a, b and m are real numbers with a and b non-zero.

(a) In this part of the question $a=2$ and $b=1$.

(i) On the axes in the Printed Answer Booklet, sketch the curve in each of these cases.

- $m=0$
- $m=1$
- $m=2$ **[3]**

(ii) State **one** feature of the curve for the case $m=0$ that is **not** a feature of the curve in the cases $m=1$ and $m=2$. **[1]**

For the remainder of this question $m=0$.

(b) Verify that the parametric equations of the curve are

$$x(t)=a\cos(t),\ y(t)=b\sin(t),$$

where $0\leqslant t<2\pi$ is a parameter. **[1]**

(c) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{b}{a}\cot(t)$. **[2]**

(d) Show that the equation of the normal to the curve at the point with parameter t is

$$y=\frac{a}{b}\tan(t)x-\left(\frac{a^2-b^2}{b}\right)\sin(t).$$ **[5]**

(e) Show that the parametric equations of the envelope of the normal to the curve are

$$x(t)=\left(\frac{a^2-b^2}{a}\right)\cos^3(t),\ y(t)=\left(\frac{b^2-a^2}{b}\right)\sin^3(t),$$

where $0\leqslant t<2\pi$ is a parameter. **[6]**

(f) In this part of the question $a=2$ and $b=1$.

On the axes in the Printed Answer Booklet, sketch the envelope of the normal to the curve. **[1]**

(g) By considering the expression $(ax(t))^{\frac{2}{3}}+(by(t))^{\frac{2}{3}}$ or otherwise, determine a cartesian equation of the envelope of the normal to the curve. **[2]**

2 **(a)** **(i)** Write down $7^{10} \pmod{1000}$. **[1]**

Fermat's little theorem states that if p is a prime and x is an integer which is co-prime to p, then $x^{p-1} \equiv 1 \pmod{p}$.

(ii) Explain why Fermat's little theorem implies $2^{12} \equiv 1 \pmod{13}$. **[1]**

(iii) Determine $2^{(12q+1)} \pmod{13}$, where q is a positive integer. **[2]**

(b) In the rest of this question the highest common factor of positive integers m and n is denoted by (m, n).

(i) Write down (354, 27). **[1]**

Euler's totient function $\varphi(n)$, where n is a positive integer, is defined to be the number of integers m with $1 \leqslant m \leqslant n$ such that $(m, n) = 1$.

For example, $\varphi(12) = 4$ since 1, 5, 7 and 11 are all co-prime with 12, but 2, 3, 4, 6, 8, 9, 10, and 12 all share a common factor greater than 1 with 12.

(ii) Create a program which returns the value of $\varphi(n)$ for a given positive integer n.
Write out your program in full in the Printed Answer Booklet. **[4]**

(iii) Use your program to find $\varphi(1000)$. **[1]**

Euler's theorem states that if a and n are co-prime positive integers, then $a^{\varphi(n)} \equiv 1 \pmod{n}$.

(iv) Determine $7^{(400r+10)} \pmod{1000}$, where r is a positive integer. **[4]**

(v) Using part **(b)(iv)**, determine the tens digit of 7^{2010}. **[2]**

(c) Suppose that $p \geqslant 3$ is a prime number and that x and y are positive integers.

By considering the equation $p^{2x} + 1 = 2^{2y}$ modulo 4, or otherwise, prove there are **no** integer solutions to the equation $p^{2x} + 1 = 2^{2y}$. **[5]**

3 This question concerns the family of differential equations

$$\frac{dy}{dx} = x - y + ae^{x}\sin(y) \qquad (**)$$

where a is a constant.

(a) In this part of the question $a = 0$.

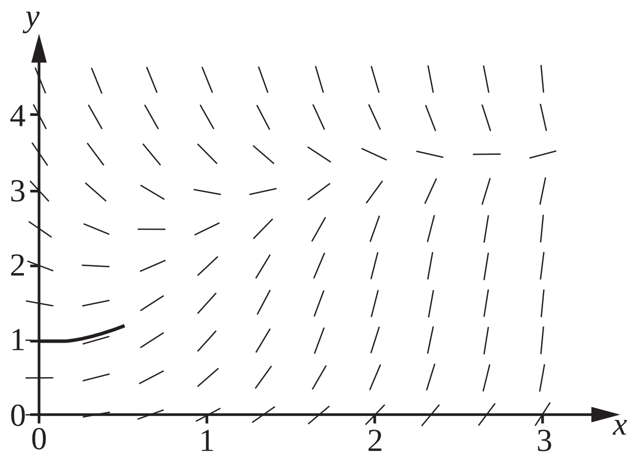
(i) Verify that $y = 2e^{-x} + x - 1$ is the particular solution of (**) that satisfies $y = 1$ when $x = 0$. **[3]**

The solution $y = 2e^{-x} + x - 1$ has a minimum value at the point (m, n) where $0 < m < 1$.

(ii) Find the exact value of m. **[2]**

(iii) Sketch the particular solution of (**) given in part **(a)(i)** for $0 \leqslant x \leqslant 3$ on the axes in the Printed Answer Booklet. **[2]**

(b) The figure below shows the tangent field for an unspecified value of a. A sketch of the solution curve $y = g(x)$ which passes through the point (0, 1) is shown for $0 \leqslant x \leqslant \frac{1}{2}$.



Continue the sketch of the solution curve for $\frac{1}{2} \leqslant x \leqslant 3$ on the axes in the **Printed Answer Booklet**. **[2]**

(c) (i) The standard Runge-Kutta method of order 4 for the solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ is as follows.

$$k_1 = h\mathrm{f}(x_n, y_n)$$

$$k_2 = h\mathrm{f}\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\mathrm{f}\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\mathrm{f}(x_n + h, y_n + k_3)$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Construct a spreadsheet to solve (**) so that the value of a and the value of h can be varied in the case $x_0 = 0$ and $y_0 = 4$. State the formulae you have used in your spreadsheet. **[4]**

(ii) In this part of the question $a = 1$, $x_0 = 0$ and $y_0 = 4$.

Use your spreadsheet with $h = 0.05$ to determine an approximation of the value of y for the solution to (**) when $x = 1$, giving your answer correct to **2** decimal places. **[1]**

(iii) In this part of the question $a = 0.5$, $x_0 = 0$ and $y_0 = 4$.

The solution to (**) has a minimum point (r, s) with $1 < r < 2$.

Use your spreadsheet with suitable values of h to determine the value of r correct to **2** decimal places. **[2]**

(iv) In this part of the question $h = 0.05$, $x_0 = 0$ and $y_0 = 4$.

There is a value of $a > 0$ such that $y(1)$ is 2.47 to 2 decimal places.

By varying the value of a in your spreadsheet, find a suitable value of a correct to **1** decimal place. **[2]**

END OF QUESTION PAPER

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